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Solvable potentials associated with $su(1,1)$ algebras: a systematic study

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Abstract. We consider a specific differential realization of the $su(1,1)$ algebra and use it to explore such algebraic structures associated with shape-invariant potentials. Our approach combines elements of various methods of solving the Schrödinger equation, such as supersymmetric quantum mechanics (or the factorization method), algebraic techniques and special-function theory. In fact, it amounts to reformulating transformations mapping the Schrödinger equation into the differential equation of orthogonal polynomials in group-theoretical terms. Our systematic study recovers a number of earlier results in a natural unified way and also leads to new findings. The procedure presented here implicitly contains a similar treatment of the compact $su(2)$ algebra as well. Possible generalizations of this approach (involving different realizations of the $su(1,1)$ algebra, other algebraic structures and larger classes of potentials) are also outlined.

1. Introduction

Exactly solvable quantum mechanical potentials have attracted much attention since the early days of quantum mechanics, and the Schrödinger equation has been solved for a large number of potentials by employing a variety of approaches. The solutions of certain one-dimensional potentials have been given by the factorization method [1, 2] and by using the powerful machinery of group theory [3, 4], for example. More recent developments which generated renewed interest in solvable potentials were the introduction of the potential-group method [5] and supersymmetric quantum mechanics (SUSYQM) [6]. Both of these approaches give simultaneous solution of a whole series of potentials each having different depth and (almost) identical bound-state energy spectra. Although the methods mentioned above usually focus on different aspects of solvable potentials, they are not independent from each other. Supersymmetric quantum mechanics, for example, has been recognized as the reformulation of the factorization method [7], which, in turn, can be considered to be an application of the Darboux transformation method of solving second-order ordinary differential equations [8]. (See, for example [9] and references therein.) In fact, most of the approaches mentioned above can be formulated in terms of 'direct' methods of solving the Schrödinger equation, i.e. by rewriting them as transformations mapping the original Schrödinger equation into the second-order differential equation of some special function of mathematical physics. Systematic studies of these transformations have been given, for example, by Bhattacharjie and Sudarshan [10] and by Natanzon [11] regarding the confluent hypergeometric and hypergeometric functions. The relation of these methods

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to certain algebraic techniques has been discussed by Cordero *et al* [12] and Wu *et al* [13, 14], for example. In an extensive survey based on the Lie theory of special functions Miller [3] also discussed this connection and linked it with the factorization method. More recently the relation between the formalism of supersymmetric quantum mechanics and that followed by the authors of [11] and [10] has been discussed [15, 16]. These studies also lead to (inequivalent) classification schemes of potentials admitting shape-invariance, a concept inspired by the SUSYQM approach [17]. The range of these potentials, which includes the most well known textbook examples for solvable quantum mechanical problems, was later found to be identical to the set of potentials obtained from the factorization method [7].

Although the intertwining relationship between various algebraic approaches, special-function theory and factorization (or SUSYQM) has been investigated extensively, most of these studies concentrated on specific potentials and/or specific aspects of this problem. In the present study we try to combine these approaches by performing a systematic search for $su(1,1)$ algebraic structures associated to shape-invariant potentials. In particular, we start from a specific differential realization of the $su(1,1)$ algebra (inspired by the potential-group method [5]), which can be used to derive the second-order differential equation of orthogonal polynomials. The second-order differential operator which appears in the differential equation of these functions is expressed in terms of the Casimir operator of the $su(1,1)$ algebra. Following Wu *et al* [13, 14] we then apply variable and similarity transformations to the group generators in order to recover the Schrödinger equation for various potentials. If these transformations leave the $su(1,1)$ algebra intact, we get solvable potentials which are automatically associated with an $su(1,1)$ algebra. This transformation procedure can be made systematic if we employ the variable transformations identified from a SUSYQM-related study which recovered shape-invariant potentials from the differential equations of orthogonal polynomials [16]. In this sense we can formulate these transformations in the language of group theory and use all the powerful machinery which is associated to it.

The relation between $su(1,1) \simeq so(2,1)$ algebraic structures and solvable potentials has been studied extensively [4, 5, 12, 18–26]. Being the elements of a non-compact algebra, the J_+ and J_- ladder operators are able to connect members of an infinite sequence of basis states. In these applications this non-compact algebra appears either as a spectrum-generating algebra [12, 18–20, 24, 25], or as a construction identified as potential algebra [5, 21, 24, 27]. (Symmetry algebras, which account for the degeneracies of a large variety of quantum mechanical systems (see, for example [28] and references therein for a review) are not expected to be relevant to the simple one-dimensional potential problems we investigate here.) The earlier works mentioned above usually focus on individual potentials, consider various realizations of the $su(1,1)$ algebra and occasionally describe more general algebras which contain $su(1,1)$ as a subalgebra. We shall discuss the relation of these results to our systematic approach, which we expect to recover a significant part of these earlier developments in a unified way.

The $su(1,1)$ ($\simeq so(2,1) \simeq sl(2,R)$) algebra is the non-compact real form of the complex semi-simple Lie algebra A_1 [4]. Its compact version $su(2)$ ($\simeq so(3)$) has also been used extensively in the description of various physical problems. (See, for example, [4, 29] and references therein.) Our approach can also be applied to this compact algebra in a straightforward way, furthermore, we shall see that it automatically leads to $su(2)$ algebras in some cases. The reason why we present the formalism within the framework of the non-compact $su(1,1)$ algebra is that our work has mainly been motivated by the potential-group method [5] (and its similarity with SUSYQM), where the non-compact nature of the algebra plays an important role.

It has to be mentioned that our present approach is not the most general one. Here we considered only the $su(1,1)$ algebra and only one possible differential realization of it. Furthermore, we consider only those transformations, which recover the trivial shape-invariant potentials from the differential equation of the orthogonal polynomials $L_n^{(\alpha)}(x)$, $H_n(x)$, $P_n^{(\alpha,\beta)}(x)$ and $C_n^{(\alpha)}(x)$. Nevertheless, this method can be generalized in various directions: further realizations and larger algebras could be taken, and transformations leading to non-shape-invariant potentials could also be considered. However, for the sake of compactness we leave these options for a further study.

The arrangement of this paper is as follows. In section 2 we discuss the relation between SUSYQM and solvable potentials. In section 3 we introduce the differential realization of the $su(1,1)$ algebra we apply later on, and also describe the effect of variable and similarity transformations on its generators. Our main results are contained in section 4, where we combine the contents of the two previous sections and use them to formulate in algebraic terms the transformation of the differential equations of orthogonal polynomials into the Schrödinger equation with shape-invariant potentials. Finally, we summarize our results in section 5.

2. Supersymmetric quantum mechanics and solvable potentials

Starting from a superalgebra of operators and their matrix realization, supersymmetric quantum mechanics offers an elegant and straightforward way of relating pairs of isospectral one-dimensional quantum mechanical potentials. (See, for example, [6] for the details.) A compact way of writing the Schrödinger equation (with $\hbar = 2m = 1$) for these supersymmetric partner potentials is

$$H_{\pm}\Psi^{(\pm)}(x) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{\pm}(x)\right)\Psi^{(\pm)}(x) = E^{(\pm)}\Psi^{(\pm)}(x) \tag{2.1}$$

where $V_-(x)$ and $V_+(x)$ are expressed in terms of the superpotential $W(x)$ as

$$V_{\pm}(x) = W^2(x) \pm \frac{\mathrm{d}W}{\mathrm{d}x}. \tag{2.2}$$

It is easy to show [6] that in the simplest case supersymmetry manifests itself in the following degeneracy of the energy levels of the supersymmetric partner potentials:

$$E_{n+1}^{(-)} = E_n^{(+)} \qquad n = 0,\ 1,\ 2,\ldots \qquad E_0^{(-)} = 0 \tag{2.3}$$

and that the degenerate eigenstates are connected by the linear differential operators

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad A^{\dagger} = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{2.4}$$

where $W(x)$ is related to the ground-state wavefunction of $V_-(x)$ [6].

An immediate practical consequence of these results is that whenever the ground-state wavefunction of a potential $V_-(x)$ is known, its supersymmetric partner $V_+(x)$ can readily be constructed, and this potential will have the same energy eigenvalues as the original one, except that its ground state will be degenerate with $E_1^{(-)}$. In fact, if all the wavefunctions of $V_-(x)$ are known, a whole series of potentials can be constructed this way, each having one less bound state than the previous one. Adjacent members of this series of potentials are supersymmetric partners.

It was soon noticed that supersymmetric partner potentials often depend on the coordinate in the same way, and differ only in some parameters which set their depth. Gendenshtein [17] defined these 'shape-invariant' potentials by the relationship

$$V_+(x,a_0) - V_-(x,a_1) \equiv W^2(x,a_0) + W'(x,a_0) - W^2(x,a_1) + W'(x,a_1) = R(a_1) \qquad (2.5)$$

where a_0 and a_1 stand for potential parameters in the supersymmetric partner potentials, and $R(a)$ is a constant. The two sets of potential parameters a_0 and a_1 are connected by simple mathematical formulae. It has been shown that the energy eigenvalues [17] and the wavefunctions [30] of shape-invariant potentials can be constructed practically without solving the Schrödinger equation, simply by utilizing the defining equation of shape invariance and the properties of the $A^\dagger$ and A operators.

Many of the well known solvable potentials of quantum mechanics were found to have shape-invariant properties. In fact, it turned out that shape-invariant potentials are exactly the same as the ones that can be obtained from the factorization method [1,2]. Several attempts have been made to find and classify all shape-invariant potentials [15,16,31], and the results sugested that besides the 12 known potentials there were no others. These studies also clarified the relationship between shape invariance and solvability [15], and identified shape-invariant potentials as special subclasses of the general Natanzon [11] and Natanzon confluent [11,25] potentials. These potentials depend on six parameters and their solutions contain hypergeometric and confluent hypergeometric functions, respectively.

We mention here that recently a new family of potentials satisfying the (2.5) shape-invariance condition has been identified [32]. These potentials have more complicated mathematical structure than the shape-invariant potentials found originally. We shall call the latter ones trivial shape-invariant potentials for further reference.

Here we use an old method of solving the Schrödinger equation to demonstrate how a wide range of solvable potentials can be recovered within the framework of supersymmetric quantum mechanics. Originally this procedure was used [10] to derive only some known potentials, but it can be proved that all the Natanzon and Natanzon confluent potentials could also have been derived from it. It was later connected to the formalism of SUSYQM [16].

The solutions of the one-dimensional Schrödinger equation (with $\hbar = 2m = 1$)

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + (E - V(x))\Psi(x) = 0 \qquad (2.6)$$

are generally written as

$$\Psi(x) = f(x)F(y(x)) \qquad (2.7)$$

where $F(y)$ is a special function which satisfies a second-order differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}y^2} + Q(y)\frac{\mathrm{d}F}{\mathrm{d}y} + R(y)F(y) = 0\,. \qquad (2.8)$$

Here $Q(y)$ and $R(y)$ are well known for any specified special function $F(y)$, while $f(x)$ and $y(x)$ are some functions to be determined. After some straightforward algebra we arrive at the following expression [16]:

$$E - V(x) = (y'(x))^2R(y(x)) - \left(\left(\frac{f'(x)}{f(x)}\right) + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f'(x)}{f(x)}\right)\right) \qquad (2.9a)$$

$$= \frac{y'''(x)}{2y'(x)} - \frac{3}{4}\left(\frac{y''(x)}{y'(x)}\right)^2 + (y'(x))^2\left(R(y(x)) - \frac{1}{2}\frac{\mathrm{d}Q(y)}{\mathrm{d}y} - \frac{1}{4}Q^2(y(x))\right)\,. \qquad (2.9b)$$

Equation (2.9b) relates the only undetermined function $y(x)$ to the difference of the energy E and the potential $V(x)$. Observing that the energy term E on the left-hand side of (2.9) represents a constant, the authors of [10] equated certain terms of the right-hand side with a constant (denoted by C for further reference) to account for it. This results in simple differential equations for $y(x)$. The authors in [10] applied this method to the hypergeometric and confluent hypergeometric function and obtained the solutions of some simple potentials.

As described in [16], equation (2.9a) offers a straightforward connection to the formalism of supersymmetric quantum mechanics. In particular, whenever $R(y)$ vanishes for the ground state, we have

$$E - V(x) = -W^2(x) + \frac{\mathrm{d}W}{\mathrm{d}x} \tag{2.10}$$

which corresponds to a potential $V_-(x)$ from (2.2) having a ground state with zero energy. In [16] this method was applied to the orthogonal polynomials ($P_n^{(\alpha,\beta)}(y)$, $L_n^{(\alpha)}(y)$ and $H_n(y)$) which, indeed, fulfill the $R_{n=0}(y) = 0$ requirement, and the results were used to derive a straightforward classification scheme of the trivial shape-invariant potentials. In addition to shape-invariant potentials, some general Natanzon class [11] potentials have also been identified [33, 34] by this method. See [34] for an example for this procedure.

3. The $SU(1, 1)$ group and solvable potentials

The generators of the non-compact $SU(1, 1)$ group obey the commutation relations

$$[J_z, J_\pm] = \pm J_\pm \tag{3.1a}$$

$$[J_+, J_-] = -2J_z \tag{3.1b}$$

and are related to the Casimir operator as

$$C_2 = -J_+J_- + J_z^2 - J_z \tag{3.2a}$$

$$= -J_-J_+ + J_z^2 + J_z\,. \tag{3.2b}$$

The eigenstates of C_2 and J_z (with eigenvalues $j(j+1)$ and m, respectively) serve as a basis for the irreducible representations of $SU(1, 1)$, and can be labelled by $|jm\rangle$.

This relatively simple mathematical construction can be used to recover a wide range of physical results. Introducing the concept of the potential group [5] Alhassid *et al* suggested that the discrete and continuous unitary irreducible representations of this group can be identified with bound and scattering states of a wide range of quantum mechanical potentials in one dimension. In this case the generators are realized in terms of linear differential operators, and the Hamiltonian is related to the Casimir operator as

$$H = -C_2 - \tfrac{1}{4}\,. \tag{3.3}$$

The physical interpretation of this group-theoretical result is that the irreducible representations of the $SU(1, 1)$ potential group are formed by states which have the same energy eigenvalues ($E = -(j + \frac{1}{2})^2$), but correspond to different potential strength, which is related to m. In particular, bound states of these potentials belong to the discrete unitary irreducible representation of the $SU(1,1)$ group called discrete principal series D_j^+, for which j can assume integer or half-integer values, while the allowed values of m are

$$m = -j,\ -j+1,\ -j+2, \dots . \tag{3.4}$$

Scattering states are assigned to continuous unitary irreducible representations of $SU(1,1)$.

These results were used to derive in a straightforward way the solution of the Morse, Pöschl–Teller and the Ginocchio potentials [5, 22]. The connection of the $SU(1,1) \simeq SO(2,1)$ and the $SO(2,2)$ potential groups to the general Natanzon class [11] potentials has also been discussed [13, 14].

Shortly after the introduction of $SU(1,1)$ as a potential group, Sukumar also used a differential realization of the $su(1,1)$ algebra to generate solvable potentials [23]. Similarly to Alhassid *et al* [5] he considered the basis states

$$|jm\rangle = \Psi_{jm}(x) = \mathrm{e}^{\mathrm{i}m\phi}\psi_{jm}(x) \tag{3.5}$$

but allowing a more general form of the generators

$$J_{\pm} = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm h(x)\frac{\partial}{\partial x} \pm g(x) + f(x)J_z + c(x)\right) \tag{3.6a}$$

and

$$J_z = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{3.6b}$$

he showed that they satisfy (3.1*b*) if

$$f^2(x) - h(x)\frac{\mathrm{d}f}{\mathrm{d}x} = 1 \tag{3.7a}$$

and

$$h(x)\frac{\mathrm{d}c}{\mathrm{d}x} - c(x)f(x) = 0 \tag{3.7b}$$

hold. (Equation (3.1*a*) is automatically satisfied by this construction.) In terms of this realization the Casimir operator has the form

$$\begin{aligned} C_2 = h^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x} + h(x)\left(\frac{\mathrm{d}h}{\mathrm{d}x} + 2g(x) - f(x)\right)\frac{\mathrm{d}}{\mathrm{d}x} \\ - \left(f(x)g(x) - g^2(x) - h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + c^2(x)\right) - 2c(x)f(x)J_z + (1 - f^2(x))J_z^2 . \end{aligned} \tag{3.8}$$

Sukumar used these results to recover the potentials discussed by Alhassid *et al* as special cases of a wider class of potentials.

An interesting way of extending the range of potentials that can be described within the potential-group formalism has also been pointed out by Wu *et al* [13, 14, 35], who noted that the $su(1,1)$ algebra (3.1) is maintained if one applies a variable or similarity transformation of the generators $J_{\pm}$, J_z and the basis $|jm\rangle$. This is easily verified within the more general expression of the generators in (3.6). In particular, a variable transformation $z = z(x)$ changes the functions $h(x)$, $g(x)$, $f(x)$, $c(x)$ and the basis states in the following way:

$$
\begin{aligned}
h(x) &\to h_v(z) = h(x(z))\frac{dz}{dx} \equiv h(x(z))u(z) \\
g(x) &\to g_v(z) = g(x(z)) \\
f(x) &\to f_v(z) = f(x(z)) \\
c(x) &\to c_v(z) = c(x(z)) \\
\Psi_{jm}(x) &\to \Psi_{vjm}(z) = \Psi_{jm}(x(z))
\end{aligned}
\tag{3.9}
$$

while a similarity transformation

$$
J_{s\alpha} = \mathcal{F} J_\alpha \mathcal{F}^{-1} \qquad (\alpha = +, \ -, \ z) \tag{3.10}
$$

where $\mathcal{F} = 1/v(x)$ represents a multiplication with a function $1/v(x)$ (written in this form for the sake of convenience) leads to the following results:

$$
\begin{aligned}
h(x) &\to h_s(x) = h(x) \\
g(x) &\to g_s(x) = g(x) + h(x)\frac{d}{dx}\ln v(x) \\
f(x) &\to f_s(x) = f(x) \\
c(x) &\to c_s(x) = c(x) \\
\Psi_{jm}(x) &\to \Psi_{sjm}(x) = \frac{1}{v(x)}\Psi_{jm}(x) .
\end{aligned}
\tag{3.11}
$$

It is obvious that whenever (3.7a) and 3.7(b) hold for the original set of equations, similar relations will hold for the transformed functions too. However, this requirement alone does not guarantee that the $su(1,1)$ algebra remains intact. If the functions governing transformations (3.9) and (3.11) (i.e. $z(x)$ and $v(x)$) depend explicitly on m, the eigenvalue of generator J_z, the commutation relations cease to be valid in the most general case, and will hold only when the corresponding operator equation is applied to basis states labelled with m. This, of course, means that the algebraic construction becomes obsolete, nevertheless, these transformations and the differential equations obtained from them may be of interest in their own right. Furthermore, besides cases in which such m-dependent transformations destroy an existing $su(1,1)$ algebra, we shall see that there are examples for the reverse mechanism, too. In the earlier applications of transformations (3.9) and (3.11) these complications have been avoided by Alhassid *et al* [22], who derived the Ginocchio potential [36] from the potential-group approach.

It can also be seen that transformations (3.9) and (3.11) can be used to transform the generators originally considered by Alhassid *et al* [5, 22] (i.e. $h(x) = 1$, $g(x) = f(x)/2$) to the more general expressions in (3.6). An important implication of selecting an $h(x)$ other than ± 1 is that one has to redefine the relation of the Casimir operator and the Hamiltonian, as the simple equations $H = -C_2 - \frac{1}{4}$ and $E = -(j + \frac{1}{2})^2$ cease to be valid then. The new form of the eigenvalue equation is

$$
\begin{aligned}
H\Psi(x) &= \left(\left(\frac{1}{h^2(x)} - d \right) \left(j + \tfrac{1}{2}\right)^2 + \frac{1}{h^2(x)} \left(-C_2 - \tfrac{1}{4}\right) \right) \Psi(x) \\
&= -d \left(j + \tfrac{1}{2}\right)^2 \Psi(x)
\end{aligned}
\tag{3.12}
$$

where d is a constant [35]. This choice really results in the usual form of the Schrödinger equation, but, as we shall see later, it may result in cases when H does not commute with J_+ and J_-, so the $su(1,1)$ algebra does not play the role of a potential algebra any more.

Finally, we mention that the mathematical construction presented in this section can be used to recover potentials associated with the compact $su(2)$ algebra as well. This requires only the redefinition of the functions $h(x)$, $f(x)$, $g(x)$ and $c(x)$ (and, therefore, also that of $J_\pm$) by multiplying them with the constant imaginary factor i (or −i). This operation changes the sign of the right-hand side of (3.7a), but does not affect (3.7b). The expression for the Casimir operator in (3.8) also remains valid in its present form: the $h(x) \rightarrow \mathrm{i}h(x), \ldots$, etc transformation simply changes the sign of its terms except for the one J_z^2.

4. Search for $su(1,1)$ algebraic structures related to solvable potentials

The two methods, SUSYQM and algebraic techniques, represent alternative ways of describing the same classes of solvable potentials. In addition to this, there are some closer analogies between SUSYQM and the potential-group method. Both approaches connect infinite series of potentials which have the same energy eigenvalues, and differ only in their depth, and therefore in the number of their bound states. The operators which connect the degenerate eigenstates of these potentials have similar structure in SUSYQM and in the potential-group method: they are both linear differential operators (see (2.4) and (3.6a)). These similarities naturally raise the question whether there exists a connection between SUSYQM and the algebraic approach. Here we try to answer this question by comparing the mathematical manipulations used in the two approaches to transform the Schrödinger equation into the second-order differential equation of some special function. For SUSYQM these techniques have been described in detail in section 2, and here we show that these transformations can be formulated within the framework of the algebraic approach too.

Let us consider a special function $F(y)$ which satisfies a second-order differential equation of the form (2.8). We can introduce a mathematical construction similar to that described in section 3, with the difference that we denote the independent variable by y rather than by x. Constructing the $SU(1,1)$ generators and the Casimir operators in the usual way we can introduce the 'null operator' [35]

$$X \equiv (C_2 - j(j+1)) \tag{4.1}$$

which we can use to rewrite the differential equation of the $F(y)$ function in terms of the Casimir operator of the $su(1,1)$ algebra (see (3.8)):

$$\begin{aligned} X\Psi_{jm}(y) = \Bigg[& h^2(y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + h(y)\left(\frac{\mathrm{d}h}{\mathrm{d}y} + 2g(y) - f(y)\right)\frac{\mathrm{d}}{\mathrm{d}y} - (1 - f^2(y))J_z^2 \\ & - f(y)g(x) + g^2(y) + h(y)\frac{\mathrm{d}g}{\mathrm{d}y} - 2c(y)f(y)J_z - c^2(y) - j(j+1)\Bigg]\Psi_{jm}(y) \\ = {} & 0 . \end{aligned} \tag{4.2}$$

After the $\langle J_z \rangle = m$ substitution and the separation of the ϕ-dependent part of the wavefunction we can easily identify the $Q(y)$ and $R(y)$ functions of (2.8).

At this point a two-step procedure can be employed to transform (4.2) into a Schrödinger equation related to an $su(1,1)$ algebra.

(i) Introduce a variable transformation $x = x(y)$.
(ii) Select a similarity transformation which eliminates the linear differential term in the Hamiltonian obtained after the first step. This can be done by an appropriate choice of $g(x)$ (see equations (3.11) and (4.2)).

These transformations yield wavefunctions of the form (2.7), which indicates that we have accomplished the same transformation as in section 2, and have obtained the same potentials as in supersymmetric quantum mechanics.

Here we apply this two-step procedure to derive the trivial shape-invariant potentials [16] within the framework of the algebraic treatment and discuss what role the $su(1,1)$ algebra plays in these cases. This requires the construction of $su(1,1)$ algebras from which the differential equation of orthogonal polynomials can be derived. The variable transformations $y \to x(y)$ then emerge in a straightforward way from the SUSYQM approach to the same (shape-invariant) potentials [16].

4.1. Generalized Laguerre polynomials $L_n^{(\alpha)}(y)$

The differential equation of the generalized Laguerre polynomials [37] can be obtained from the 'null operator' (4.2) with the following choice of the generators J_+ and J_- and of the wavefunctions:

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm y\frac{\partial}{\partial y} \pm \frac{1}{2}(\alpha + 1 - y) + J_z - \frac{y}{2}\right) \tag{4.3a}$$

$$\Psi_{jm}(y) = \mathrm{e}^{\mathrm{i}m\phi}L_n^{(\alpha)}(y) \tag{4.3b}$$

where j and m are related to α and n in the following way:

$$\alpha = -(2j+1)$$
$$m = -j + n \qquad n = 0, 1, 2, \ldots . \tag{4.4}$$

The appropriate variable transformation in the LIII case is defined by $y(x) = b\exp(-ax)$, where $a = C^{1/2}$ [16]. Performing the similarity transformation used in the second step to recover the Schrödinger equation we find that the resulting potential is the Morse potential. The potential, the energy eigenvalues and the (unnormalized) wavefunctions are displayed in table 1, while tables 2 and 3 show the $J_\pm$ generators, the SUSYQM laddering operators A, $A^\dagger$ and their effect on the wavefunctions.

It can be seen from tables 2 and 3 that the J_+, J_- and $A^\dagger$, A operators are essentially the same, as they have the same effect on the wavefunctions, and furthermore the $su(1,1)$ algebra is the well known potential algebra [5] for the Morse potential. The infinite sequence of isospectral potentials described by both the potential-group method and supersymmetric quantum mechanics are Morse potentials which have different depth due to parameter m. A minor difference between the two sets of operators arises because of the presence of the phase factors $\mathrm{e}^{\pm \mathrm{i}\phi}$ and $\mathrm{e}^{\mathrm{i}m\phi}$ in the algebraic construction. We mention here that essentially the same results were also obtained from a different differential realization of the $su(1,1)$ algebra [21, 24].

Similar treatment of the LI case [16] recovers the harmonic oscillator potential in three dimensions. (See table 1, where the notation $\alpha = l + \frac{1}{2}$ and (4.4) has been used.) In contrast with the Morse potential, now the $su(1,1)$ and SUSYQM ladder operators (in tables 2 and 3) are essentially different. A and $A^\dagger$ connect states with angular momenta differing by one unit. These states, of course, can not be degenerate in the usual description, only in

Table 1. *The final form of the potential, energy eigenvalues and (unnormalized) wavefunctions obtained from the differential realization of the* $su(1,1)$ *algebra. The potentials are labelled according to the notation of [16]. (The five* PI *class potentials there correspond to the following choice of the parameters here:* $m = s+\frac{1}{2}$, $s+\frac{1}{2}$, $-s+\frac{1}{2}$, $-\frac{1}{2}(\lambda+s-1)$, $-\frac{1}{2}(\lambda-s-1)$ *and* $b = -\lambda$, $-\mathrm{i}\lambda$, $\mathrm{i}\lambda$, $\mathrm{i}\frac{1}{2}(s-\lambda)$, $-\mathrm{i}\frac{1}{2}(\lambda+s)$.) *For the sake of brevity here we abandoned the* $E_0 = 0$ *requirement of* SUSYQM.

Class	$y(x)$	C	$V(x)$
LIII	$b\exp(-ax)$	a^2	$\frac{Cb^2}{4}\exp(-2ax) - Cb\exp(-ax)$
LI	$\frac{1}{2}\omega x^2$	2ω	$\frac{\omega^2}{4}x^2 + \frac{l(l+1)}{x^2}$
HI	$\left(\frac{\omega}{2}\right)^{1/2}x$	$\frac{\omega}{2}$	$\frac{\omega}{4}x^2$
PI			$-C\frac{b^2-(m-\frac{1}{2})(m+\frac{1}{2})+2\mathrm{i}bmy(x)}{1-y^2(x)}$
	$\mathrm{i}\sinh(ax)$	$-a^2$	$a^2\frac{b^2-(m-\frac{1}{2})(m+\frac{1}{2})-2bm\sinh(ax)}{\cosh^2(ax)}$
	$\cosh(ax)$	$-a^2$	$a^2\frac{b^2-(m-\frac{1}{2})(m+\frac{1}{2})+2\mathrm{i}bm\cosh(ax)}{\sinh^2(ax)}$
	$\cos(ax)$	a^2	$-a^2\frac{b^2-(m-\frac{1}{2})(m+\frac{1}{2})+2\mathrm{i}bm\cos(ax)}{\sin^2(ax)}$
	$\cos(2ax)$	$4a^2$	$a^2\frac{(\mathrm{i}b+m-\frac{1}{2})(\mathrm{i}b+m+\frac{1}{2})}{\cos^2(ax)} + a^2\frac{(-\mathrm{i}b+m-\frac{1}{2})(-\mathrm{i}b+m+\frac{1}{2})}{\sin^2(ax)}$
	$\cosh(2ax)$	$-4a^2$	$-a^2\frac{(\mathrm{i}b+m-\frac{1}{2})(\mathrm{i}b+m+\frac{1}{2})}{\cosh^2(ax)} + a^2\frac{(-\mathrm{i}b+m-\frac{1}{2})(-\mathrm{i}b+m+\frac{1}{2})}{\sinh^2(ax)}$
PII			$-Cj(j+1)(1-y^2(x))$
	$\tanh(ax)$	a^2	$-a^2\frac{j(j+1)}{\cosh^2(ax)}$
	$-\mathrm{i}\cot(ax)$	$-a^2$	$a^2\frac{j(j+1)}{\sin^2(ax)}$
Class	$y(x)$	E_n	$\psi_n(x)$
LIII	$b\exp(-ax)$	$-a^2\left(m-\frac{1}{2}-n\right)^2$	$\exp(-(m-\frac{1}{2}-n)ax)\exp\left(-\frac{b}{2}\exp(-ax)\right)$ $\times L_n^{(2m-1-2n)}b\exp(-ax)$
LI	$\frac{1}{2}\omega x^2$	$(2n+l+\frac{3}{2})\omega$	$\exp\left(-\frac{\omega}{4}x^2\right)x^{l+1}L_n^{(l+1/2)}(\omega x^2/2)$
HI	$\left(\frac{\omega}{2}\right)^{\frac{1}{2}}x$	$\omega\left(n+\frac{1}{2}\right)$	$\exp\left(-\frac{\omega}{4}x^2\right)H_n\left(\left(\frac{\omega}{2}\right)^{1/2}x\right)$
PI		$C\left(m-\frac{1}{2}-n\right)^2$	$\left(\frac{1-y(x)}{1+y(x)}\right)^{\mathrm{i}b/2}(1-y^2(x))^{-\frac{1}{2}(m-\frac{1}{2})}P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}y(x)$
	$\mathrm{i}\sinh(ax)$	$-a^2\left(m-\frac{1}{2}-n\right)^2$	$(\cosh(ax))^{-(m-\frac{1}{2})}\exp\left(b\tan^{-1}(\sinh(ax))\right)$ $\times P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}(\mathrm{i}\sinh(ax))$
	$\cosh(ax)$	$-a^2\left(m-\frac{1}{2}-n\right)^2$	$\left(\frac{1-\cosh(ax)}{1+\cosh(ax)}\right)^{\mathrm{i}b/2}(\sinh(ax))^{-(m-\frac{1}{2})}$ $\times P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}(\cosh(ax))$
	$\cos(ax)$	$a^2\left(m-\frac{1}{2}-n\right)^2$	$\left(\frac{1-\cos(ax)}{1+\cos(ax)}\right)^{\mathrm{i}b/2}(\sin(ax))^{-(m-\frac{1}{2})}P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}(\cos(ax))$
	$\cos(2ax)$	$4a^2\left(m-\frac{1}{2}-n\right)^2$	$(\sin(ax))^{\mathrm{i}b-m+\frac{1}{2}}(\cos(ax))^{-\mathrm{i}b-m+\frac{1}{2}}$ $\times P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}(\cos(2ax))$
	$\cosh(2ax)$	$-4a^2\left(m-\frac{1}{2}-n\right)^2$	$(\sinh(ax))^{\mathrm{i}b-m+\frac{1}{2}}(\cosh(ax))^{-\mathrm{i}b-m+\frac{1}{2}}$ $\times P_n^{(\mathrm{i}b-m,-\mathrm{i}b-m)}(\cosh(2ax))$
PII		$-C(j-n)^2$	$(1-y^2(x))^{-\frac{1}{2}(n-j)}P_n^{(j-n,j-n)}(y(x))$
	$\tanh(ax)$	$-a^2(j-n)^2$	$(\cosh(ax))^{n-j}P_n^{(j-n,j-n)}(\tanh(ax))$
	$-\mathrm{i}\cot(ax)$	$a^2(j-n)^2$	$(\sin(ax))^{n-j}P_n^{(j-n,j-n)}(-\mathrm{i}\cot(ax))$

the SUSYQM approach, where harmonic oscillator states with different l constitute different potentials connected pairwise by supersymmetry, and the energy scales are shifted by one unit (1ω). The J_+ and J_- generators leave the angular momentum intact, and connect states with different number of nodes (n). This operation changes the energy by two units (2ω), so the $su(1,1)$ algebra can not be a potential algebra in this case. This result is related to the fact that $h(x) = x/2 \neq$ constant in this case, so the Hamiltonian has the more general form of (3.12), and does not necessarily commute with the generators. It, in fact, does not commute with the generators in this case, because the constant d in (3.12) depends on m, the eigenvalue of J_z.

Table 2. Explicit form of the generators J_- and J_+ and their effect on the wavefunctions.

Class	$y(x)$	$J_\pm$	Effect of J_+ and J_-
LIII	$b\exp(-ax)$	$\frac{1}{a}e^{\pm i\phi}\left(\pm\frac{\partial}{\partial x} - a\left(J_z \pm \frac{1}{2}\right) + \frac{ab}{2}\exp(-ax)\right)$	$J_+\psi_n(m;x) \to \psi_{n+1}(m+1;x)$ $J_-\psi_n(m;x) \to \psi_{n-1}(m-1;x)$
LI	$\frac{1}{2}\omega x^2$	$e^{\pm i\phi}\left(\pm\frac{1}{2}x\frac{\partial}{\partial x} + J_z \pm \frac{1}{4} - \frac{\omega}{4}x^2\right)$	$J_+\psi_n(l;x) \to \psi_{n+1}(l;x)$ $J_-\psi_n(l;x) \to \psi_{n-1}(l;x)$
HI	$\left(\frac{\omega}{2}\right)^{1/2}x$	$e^{\pm i\phi}\left(\pm\frac{x}{2}\frac{\partial}{\partial x} \pm \frac{1}{4} + J_z - \frac{\omega}{4}x^2\right)$	$J_+\psi_n(x) \to \psi_{n+2}(x)$ $J_-\psi_n(x) \to \psi_{n-2}(x)$
PI		$\frac{ie^{\pm i\phi}}{C^{1/2}}\left(\pm\frac{\partial}{\partial x} + \frac{C^{1/2}}{(1-y^2(x))^{1/2}}\left(y(x)\left(J_z \pm \frac{1}{2}\right) - ib\right)\right)$	$J_+\psi_n(m,b;x) \to \psi_{n+1}(m+1,b;x)$ $J_-\psi_n(m,b;x) \to \psi_{n-1}(m-1,b;x)$
	$i\sinh(ax)$	$\frac{1}{a}e^{\pm i\phi}\left(\pm\frac{\partial}{\partial x} - a\tanh(ax)\left(J_z \pm \frac{1}{2}\right)\right.$ $\left.+ab\,\mathrm{sech}(ax)\right)$	
	$\cosh(ax)$	$-\frac{1}{a}e^{\pm i\phi}\left(\pm\frac{\partial}{\partial x} - a\coth(ax)\left(J_z \pm \frac{1}{2}\right)\right.$ $\left.+iba\,\mathrm{cosech}(ax)\right)$	
	$\cos(ax)$	$-\frac{i}{a}e^{\pm i\phi}\left(\pm\frac{\partial}{\partial x} - a\cot(ax)\left(J_z \pm \frac{1}{2}\right)\right.$ $\left.+iba\,\mathrm{cosec}(ax)\right)$	
	$\cos(2ax)$	$-\frac{i}{2a}e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - a\cot(ax)\left(J_z \pm \frac{1}{2} - ib\right)\right.$ $\left.+a\tan(ax)\left(J_z \pm \frac{1}{2} + ib\right)\right]$	
	$\cosh(2ax)$	$-\frac{1}{2a}e^{\pm i\phi}\left[\pm\frac{\partial}{\partial x} - a\coth(ax)\left(J_z \pm \frac{1}{2} - ib\right)\right.$ $\left.-a\tanh(ax)\left(J_z \pm \frac{1}{2} + ib\right)\right]$	
PII		$\frac{ie^{\pm i\phi}}{C^{1/2}(1-y^2(x))}^{1/2}\left(\pm\frac{\partial}{\partial x} + C^{1/2}y(x)J_z\right)$	$J_+\psi_n(j;x) \to \psi_{n+1}(j;x)$ $J_-\psi_n(j;x) \to \psi_{n-1}(j;x)$
	$\tanh(ax)$	$\frac{i}{a}e^{\pm i\phi}\cosh(ax)\left(\pm\frac{\partial}{\partial x} + a\tanh(ax)J_z\right)$	
	$-i\cot(ax)$	$-\frac{1}{a}e^{\pm i\phi}\sin(ax)\left(\pm\frac{\partial}{\partial x} + a\cot(ax)J_z\right)$	

We mention here that the $su(1,1)$ algebra we have obtained is similar to (although not identical with) the one discussed by Armstrong [18], who considered a two-dimensional realization of this algebra inspired by the work of Miller [3]. Considering yet another differential realization, Cooper also arrived at similar results. (See [24] and references therein.) We also note that, in contrast with the usual case, j and m are now not integers or half-integers, rather they are quarter-integers (see (4.4) and $\alpha = l + \frac{1}{2}$).

The LII case [16] represents an interesting situation. The simple $y(x) = ax$ variable transformation (with $a = C^{1/2}$) and the corresponding similarity transformation results in the $su(1,1)$ ladder operators

$$J_\pm = e^{\pm i\phi}\left(\pm x\frac{\partial}{\partial x} + J_z - \frac{a}{2}x\right). \tag{4.5}$$

The standard procedure, in principle, results in the Coulomb potential, however, this requires the substitution $a = e^2/m$, which introduces an m-dependent variable transformation ($y(x) = (e^2/m)x$ in this case), and this destroys the $su(1,1)$ algebra, as we have seen in section 3. In other words, the problem can only be formulated separately for each value

Table 3. Explicit form of the SUSY ladder operators $A^\dagger$ and A, and their effect on the wavefunctions. The operators are displayed in their most general form, even when $J_\pm$ could not be obtained (LII class) or could be obtained for restricted cases only (PII class, for $b = 0$).

Class	$y(x)$	$A^\dagger$ (upper sign), A (lower sign)	Effect of $A^\dagger$ and A
LIII	$b\exp(-ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}\right) - \frac{ab}{2}\exp(-ax)$	$A^\dagger\psi_{n-1}(m-1;x) \rightarrow \psi_n(m;x)$ $A\psi_n(m;x) \rightarrow \psi_{n-1}(m-1;x)$
LI	$\frac{1}{2}\omega x^2$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} - \frac{l+1}{x} + \frac{\omega}{2}x$	$A^\dagger\psi_{n-1}(l+1;x) \rightarrow \psi_n(l;x)$ $A\psi_n(l+1;x) \rightarrow \psi_{n-1}(l+1;x)$
LII	$\frac{e^2}{m}x$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} - \frac{l+1}{x} + \frac{e^2}{2(l+1)}x$	$A^\dagger\psi_{n-1}(l+1;x) \rightarrow \psi_n(l;x)$ $A\psi_n(l;x) \rightarrow \psi_{n-1}(l+1;x)$
HI	$\left(\frac{\omega}{2}\right)^{1/2}x$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\omega}{2}x$	$A^\dagger\psi_{n-1}(x) \rightarrow \psi_n(x)$ $A\psi_n(x) \rightarrow \psi_{n-1}(x)$
PI		$\mp\frac{\mathrm{d}}{\mathrm{d}x} - \frac{C^{1/2}}{(1-y^2(x))^{1/2}}\left(\left(m-\frac{1}{2}\right)y(x) - \mathrm{i}b\right)$	$A^\dagger\psi_{n-1}(m-1,b;x) \rightarrow \psi_n(m,b;x)$ $A\psi_n(m,b;x) \rightarrow \psi_{n-1}(m-1,b;x)$
	$\mathrm{i}\sinh(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}\right)\tanh(ax) - ab\,\mathrm{sech}(ax)$	
	$\cosh(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}\right)\coth(ax) - \mathrm{i}ba\,\mathrm{cosech}(ax)$	
	$\cos(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}\right)\cot(ax) - \mathrm{i}ba\,\mathrm{cosec}(ax)$	
	$\cos(2ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}-\mathrm{i}b\right)\cot(ax)$ $- a\left(m-\frac{1}{2}+\mathrm{i}b\right)\tan(ax)$	
	$\cosh(2ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(m-\frac{1}{2}-\mathrm{i}b\right)\coth(ax)$ $+ a\left(m-\frac{1}{2}+\mathrm{i}b\right)\tanh(ax)$	
PII		$\mp\frac{\mathrm{d}}{\mathrm{d}x} + C^{1/2}\left(jy(x) - \mathrm{i}\frac{bm}{j}\right)$	$A^\dagger\psi_{n-1}(j-1;x) \rightarrow \psi_n(j;x)$ $A\psi_n(j;x) \rightarrow \psi_{n-1}(j-1;x)$
	$\tanh(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(j\tanh(ax) - \mathrm{i}\frac{bm}{j}\right)$	
	$\coth(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(j\coth(ax) - \mathrm{i}\frac{bm}{j}\right)$	
	$-\mathrm{i}\cot(ax)$	$\mp\frac{\mathrm{d}}{\mathrm{d}x} + a\left(j\cot(ax) + \frac{bm}{j}\right)$	

of m, and the ladder operators in (4.5) (which explicitly depend on m through a themselves) can not connect states with different principal quantum number $m = n + l + 1$. Since this contradicts the ideas on which the whole algebraic construction is based, we conclude that the $su(1,1)$ algebra is not suitable handling this problem in its present form. We note here, however, that an $su(1,1)$ algebra can be constructed for the radial Coulomb problem too, by considering a modification of the generators [19, 24]. In these cases the change of the principal quantum number (in our notation m) has to be taken care of separately.

The SUSY ladder operators (shown in table 3) change n and l by one unit and leave m invariant at the same time. Similarly to the harmonic oscillator (LI) case, the Coulomb potential represents a separate problem for each angular momentum l, and states with adjacent values of l and the node number are connected by supersymmetry.

4.2. Hermite polynomials $H_n(y)$

The differential equation of the Hermite polynomials [37] is obtained from the following choice of the $SU(1,1)$ generators:

$$J_\pm = \mathrm{e}^{\pm\mathrm{i}\phi}\left(\pm\frac{y}{2}\frac{\partial}{\partial y} \pm \left(\frac{1}{4} - \frac{y^2}{2}\right) + J_z - \frac{y^2}{2}\right) \tag{4.6}$$

if we prescribe the relations

$$(j + \tfrac{1}{4})(j + \tfrac{3}{4}) = 0 \tag{4.7a}$$

$$2n = 4(m - \tfrac{1}{4}) . \tag{4.7b}$$

It follows from (4.7) that j can be either $-\frac{1}{4}$ or $-\frac{3}{4}$, which implies that (due to $m = -j+p$) n is equal to $2p$ and $2p+1$ in these two cases, respectively. (Here we reserved n for labelling the Hermite polynomials and denoted the non-negative integer $m - j$ with p, in order to avoid confusion.)

Applied to the HI case [16], the standard procedure recovers the harmonic oscillator potential in one dimension (see table 1). Tables 2 and 3 contain the appropriate form of the $SU(1,1)$ and SUSYQM ladder operators, respectively. The two sets of operators are obviously different, and have different effects on the wavefunctions. The $SU(1,1)$ generators change n by two units, which, of course, means that they ladder within the same $SU(1,1)$ representation, and that the group generators $J_\pm$, J_z form a spectrum-generating algebra. The even and odd oscillator wavefunctions belong to the two infinite-dimensional irreducible representations labelled by $j = -\frac{1}{4}$ and $j = -\frac{3}{4}$, respectively. Similarly to the three-dimensional harmonic oscillator, these labels are quarter-integers. We note that similar results have been obtained in earlier works too, in which, however, different realizations of the $su(1,1)$ algebra were considered [38] for the one-dimensional harmonic oscillator. In contrast with J_+ and J_- the SUSYQM ladder operators connect states belonging to different parity.

4.3. Jacobi polynomials $P_n^{(\alpha,\beta)}(y)$

The differential equation of the Jacobi polynomials [37] can be obtained from (4.2) substituting

$$\begin{aligned} h(y) &= \mathrm{i}(1-y^2)^{1/2} \qquad & g(y) &= \mathrm{i}(my - \mathrm{i}b)(1-y^2)^{-1/2} \\ f(y) &= \mathrm{i}y(1-y^2)^{-1/2} \qquad & c(y) &= b(1-y^2)^{-1/2} \end{aligned} \tag{4.8a}$$

(which satisfy the equivalent of (3.7a), (3.7b)),

$$\Psi_{jm}(y) = \mathrm{e}^{\mathrm{i}m\phi} P_n^{(\alpha,\beta)}(y) \tag{4.8b}$$

and

$$\begin{aligned} \alpha &= \mathrm{i}b - m \qquad & \beta &= -\mathrm{i}b - m \\ m &= -j + n \qquad & n &= 0, 1, 2, \ldots . \end{aligned} \tag{4.9}$$

Note, however, that as $g(y)$ depends explicitly on m, the functions in (4.8a) can not be used to construct an $su(1,1)$ algebra similar to that in (3.6). Nevertheless, as we have discussed in connection with (3.9) and (3.11) it is possible to recover an $su(1,1)$ algebra via suitable transformations which eliminate the m-dependence of the generators.

It has to be mentioned though, that this $su(1,1)$ algebra constructed this way is not large enough to describe the Jacobi polynomials (or hypergeometric functions) in their most general form. As we shall see, this lack of generality manifests itself in this case in the fact that only the difference of α and β can be changed by the laddering operators which change m; the other parameter, b, which is related to the sum of α and β maintains its initial value.

The five shape-invariant potentials belonging to the PI class [16] can be generated from the solutions of the

$$\frac{(y')^2}{1-y^2} = C \tag{4.10}$$

differential equation, and with the appropriate similarity transformation (3.11). This transformation eliminates the explicit m-dependence of $g(x)$, therefore the operators built up using the resulting functions can fulfil the $su(1,1)$ commutation relations. These generators are displayed in table 2 both in the general form (with unspecified $y(x)$) and for the five individual subcases, while the SUSYQM ladder operators A, $A^\dagger$ are shown in table 3.

It is apparent from tables 2 and 3 that the $SU(1,1)$ generators J_+ and J_- have basically the same effect on the wavefunctions as the ladder operators $A^\dagger$ and A of SUSYQM; in fact, they are essentially the same, apart from some phase factors. This means that the level degeneracy of the infinite sequences of potentials in this class can be interpreted in terms of the existence of a potential group, but it can also be attributed to supersymmetry.

We mention here that in the original formulation of the $SU(1,1)$ potential group [5] only a restricted version of the first two potentials has been discussed (with $b = 0$), and the more general version was obtained only later by Sukumar [23] and Engelfield and Quesne [26].

The fifth (i.e. the Pöschl–Teller) potential has also been discussed within the framework of the potential-group method [13], but in that case the potential algebra was $so(2,2)$, which contains $su(1,1) \simeq so(2,1)$ as a subalgebra. This more general algebraic structure allows laddering between a wider range of potentials by changing the strength of the two terms of the potential independently, while in the restricted case the $SU(1,1)$ generators are able to change these parameters in a correlated way only. We mention here that this particular potential has also been obtained from a more general realization of the $su(1,1)$ algebra in [39], where the relation of this approach to geometric phases has also been discussed.

The third and fourth of the PI class potentials, which are the trigonometric versions of the second and fifth, respectively, represent an interesting situation. In contrast with the three other examples, these potentials are confined to a limited domain of space, on the borders of which they go to infinity, and their energy eigenvalues are not limited from above, rather they have a lower limit. We also see that the generators J_+ and J_- have an overall imaginary factor i in these two cases. If we absorb this into the functions $h(x)$, $g(x)$, $f(x)$ and $c(x)$ in (3.6a) we find that the group which J_+, J_- and J_z generate is no longer the non-compact $SU(1,1)$ group, rather it is the compact $SU(2)$ group. This corresponds to the change of sign on the right-hand side of (3.1b), which is the result of the change of the sign on the right-hand side of (3.7a), due to the absorption of i into the $h(x), \ldots, c(x)$ functions. At the same time, the Casimir operator basically changes only by a $\mathrm{i}^2 = -1$ factor (see (3.8)). This explains the different structure of eigenvalues, i.e. the fact that the energy eigenvalues are bounded from below, rather than from above.

The marked differences between the two kinds of these potentials can be interpreted in a straightforward way in terms of the difference between the unitary irreducible representations of compact and non-compact groups. One apparent difference is that potentials having the compact $SU(2)$ group as a potential group have no scattering states. This is natural in view of the structure (as described above) of these potentials, but in terms of group theory it is related to the fact that compact groups have no continuous unitary irreducible representations. One further difference between the two potential types arises from the finite dimensionality of the unitary irreducible representations of the $SU(2)$ group (and of compact groups, in general). This means that for a given j, i.e. for a given energy eigenvalue, potential strengths with only $|m| \leqslant j$ are allowed. Equation (4.9) also has to be modified slightly in this case.

We mention here that (up to a similarity transformation) the third case (PI($\cos(ax)$)) contains the spherical harmonics and the $so(3)$ angular momentum algebra as a special case ($b = 0$), and the compact potential group can also be recognized as the rotation group in this case. Furthermore, the formalism can be generalized to describe rotations in terms of the Euler angles by introducing a new phase (in fact, angle) variable, and by modifying the last term ($c(x)$) in $J_{\pm}$ in table 2. In this case the wavefunctions can be looked upon as matrix elements of the rotational matrices $D^{j}_{m'm}(\alpha, \beta, \gamma)$, and the physical problem can be interpreted as that of the rotating symmetric top (see, for example, [40]). There is, of course, no potential present in this case, rather the corresponding terms in the Schrödinger equation are of purely kinematical origin. The originally one-dimensional problem becomes a three-dimensional one, and the degeneracy of the energy levels can be explained by the rotational symmetry, rather than by the presence of the $SU(2)$ potential group.

When we apply the standard procedure to the PII case defined by the

$$\frac{(y')^2}{(1-y^2)^2} = C \tag{4.11}$$

differential equation [16] we find that, although it is possible, in principle, to construct the $SU(1,1)$ ladder operators

$$J_{\pm} = \frac{\mathrm{i}C^{-1/2}\mathrm{e}^{\pm\mathrm{i}\phi}}{(1-y^2(x))^{1/2}}\left(\pm\frac{\partial}{\partial x} + C^{1/2}y(x)J_z - C^{1/2}\mathrm{i}b\right) \tag{4.12}$$

and the general expression for PII class potentials, a situation similar to that in the Coulomb (LII) case arises. This is because parameter b is forced to assume m-dependence, and this destroys the $su(1,1)$ algebra. This can be avoided only if we restrict this treatment to the $b = 0$ case from the beginning. This, of course, results in less general potentials too. We have displayed the actual formulae of $V(x)$, E_n and $\Psi_n(x)$ for the restricted potentials with the general (unspecified) solution $y(x)$ of (4.11), and also for $y(x) = \tanh(ax)$ and $-\mathrm{i}\cot(ax)$ in table 1.

These two potentials are nothing but special (symmetric) versions of some PI class potentials, however the algebraic structure associated with them is different from the $su(1,1)$ and $su(2)$ potential algebras discussed there. As can be seen from table 2, J_+ and J_- in this case leave the potential (j) invariant and change n by one unit, i.e. they connect adjacent states of the same potential, therefore we conclude that they form a spectrum-generating algebra together with J_z. Similarly to the PI case, here we have two different potential types: one which has finite depth and finite number of bound states, and one which is infinitely deep and has an infinite number of bound states. This duality manifests itself in the nature of the spectrum-generating algebra too: due to the constant i factor it is the compact $su(2)$ algebra for the first potential ($y(x) = \tanh(ax)$) which has a finite number of bound states, while it is the non-compact $su(1,1)$ algebra for the second one ($y(x) = -\mathrm{i}\cot(ax)$), which has infinite number of them. In the first case the spectrum-generating algebra is practically the same as that discussed by Engelfield [41].

Note that while, in the PI case, j was related to the energy and m to the potential strength, here their roles have been exchanged.

We mention here that the third of the shape-invariant PII class potentials (with $y(x) = \coth(ax)$) is missing from our treatment, because the $b = 0$ restriction cancels the only attractive term, therefore this potential has no bound states in the restricted case. This repulsive potential can be obtained as a special limiting case from two of the PI class potentials, therefore it is possible to describe its scattering states within the framework of the non-compact $SU(1,1)$ potential group.

In contrast with the PI case, the SUSY ladder operators A and $A^{\dagger}$ essentially differ from J_- and J_+. In fact, they are the same as the SUSY ladder operators associated to PI potentials restricted to the $b = 0$ case. (Note that $V(x)$ and E_n have essentially the same form, while the wavefunctions seem to be different. This is, of course, not the case, as the two expressions are only alternative ways of writing the same functions see table 1). We also note that A and $A^\dagger$ can, of course, be constructed for the general PII class potentials too with $b \neq 0$, which was found inaccessible for the present algebraic structure.

Before closing this section we briefly mention the main results obtained for Gegenbauer polynomials $C_n^{(\alpha)}(y)$, which are simply proportional to the Jacobi polynomials [37]

$$P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(y) .$$

The differential equation of the Gegenbauer polynomials can be obtained from the following differential realization of J_+ and J_-:

$$J_\pm = \mathrm{e}^{\pm \mathrm{i}\phi}\left(\pm(1-y^2)\frac{\partial}{\partial y} \mp \alpha y \mp y J_z\right) \tag{4.13}$$

considering also the relations

$$\begin{aligned} j &= -\alpha \\ m &= -j + n \qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{4.14}$$

As expected, the standard procedure recovers those three symmetric ($b = 0$) potentials, which form a simultaneous subset of the PI and PII potential classes. An interesting observation is, however, that applying the variable transformations defining the PI class, we get spectrum-generating algebras recovered earlier in connection with PII class potentials. Also, conversely, we get the restricted version ($b = 0$) of the potential algebra related to PI-type potentials if we employ here the variable transformations defining the PII potential class. It was this restricted potential algebra which was first constructed by Alhassid *et al* [5], and it was only realized later [23, 26] that it can be extended to the more general PI class potentials by introducing a new (non-zero $c(x)$) term in J_+ and J_-. This more general algebra, of course, can not be derived from (4.13) by the usual variable and similarity transformations.

5. Summary and conclusions

Here we have made a systematic search for $su(1,1)$ algebraic structures related to shape-invariant potentials by using a specific differential realization of the $SU(1,1)$ generators. In particular, the generators were chosen as linear differential operators depending on two variables. This realization was inspired by the work of Alhassid *et al* on the potential-group approach [5], in which the Hamiltonian is expressed in terms of the Casimir operator of the potential algebra. Noting that there are formal analogies between this approach and supersymmetric quantum mechanics, we applied transformations to derive the Schrödinger equation for the trivial shape-invariant potentials from the differential equation of orthogonal polynomials, within the framework of group theory. The results are summarized in table 4. The algebras derived from the procedure described above turned out to play the role of a potential algebra or that of a spectrum-generating algebra. Whenever it was possible, we

Table 4. Summary of $su(1,1)$ and $su(2)$ algebras related to the trivial shape-invariant potentials, and their comparison with the corresponding SUSYQM laddering operators. Works in which algebras with differential realizations similar (or closely related) to that considered in our approach are also cited.

Class	$y(x)$	Potential algebra (PA)	Spectrum generating algebra (SGA)	SUSYQM operators A, $A^\dagger$
LIII	$b\exp(-ax)$	$su(1,1)$ [3,5,26,24]	—	similar to $J_\pm$ of PA
LI	$\frac{1}{2}\omega x^2$	—	$su(1,1)$ [18]	different from $J_\pm$ of SGA
LII	$\frac{e^2}{m}x$	—	—	exist
HI	$\left(\frac{\omega}{2}\right)^{1/2}x$	—	$su(1,1)$ [38,42]	different from $J_\pm$ of SGA
PI				
	$\mathrm{i}\sinh(ax)$	$su(1,1)$ [23, 26]	$su(2)$, if $b=0$ [41,26]	similar to $J_\pm$ of PA
	$\cosh(ax)$	$su(1,1)$ [26]	no bound states for $b=0$	similar to $J_\pm$ of PA
	$\cos(ax)$	$su(2)$	$su(1,1)$, if $b=0$	similar to $J_\pm$ of PA
	$\cos(2ax)$	$su(2)$	$su(1,1)$, if $b=0$	similar to $J_\pm$ of PA
	$\cosh(2ax)$	$su(1,1)$ [39]	no bound states for $b=0$	similar to $J_\pm$ of PA
PII				
	$\tanh(ax)$	$b=0$, see PI($\mathrm{i}\sinh(ax)$)	$b=0$, see PI($\mathrm{i}\sinh(ax)$)	similar to $J_\pm$ of PA for $b=0$
	$\coth(ax)$	$b=0$, see PI($\cosh(ax)$)	no bound states for $b=0$	similar to $J_\pm$ of PA for $b=0$
	$-\mathrm{i}\cot(ax)$	$b=0$, see PI($\cos(ax)$)	$b=0$, see PI($\cos(ax)$)	similar to $J_\pm$ of PA for $b=0$

also cited in table 4 earlier works describing these (or similar) algebras. We also gave a comparison of the ladder operators J_+, J_- and the SUSYQM ladder operators $A^\dagger$, A.

Potential algebras have been recovered for two classes of shape-invariant potentials: the LIII class (i.e. the Morse potential) and the PI class which contains five individual potentials. A characteristic feature of these potentials is that the variable transformation $y \to y(x)$ in (3.9) yields a constant $h(x)$ function in the generators J_+ and J_- (see (3.6a)). This then results in a Casimir operator and a 'null-operator' which is related to the Hamiltonian in a relatively simple way, and immediately recovers the Schrödinger equation. The six potentials mentioned here contain the Morse potential and the modified Pöschl–Teller potential, the classical examples for the potential-group approach [5]. Some more general potentials admitting an $SU(1,1)$ potential group [23,26] are also among those listed here. A remarkable finding is that the potential algebra is the compact $su(2) \simeq so(3)$ algebra in two cases. This is related to the fact that $h(x)$ is an imaginary constant in these cases. These two potentials have no scattering states, which can be interpreted in a straightforward way in terms of the compactness of the potential algebra. One of these cases can also be related to rotations in three spatial dimensions, and the corresponding compact potential algebra can be viewed as the angular momentum algebra.

LIII- and PI-class potentials are the same as type B and A potentials of the factorization method [2] and of the study by Miller [3] based on the Lie theory of special functions. In this latter work the $\mathcal{G}(1,0) \simeq sl(2) \oplus \mathcal{E}$ algebra is associated to these potential classes, which can be considered a generalization of the algebras discussed here.

Spectrum-generating algebras have been recovered for the LI, HI classes (i.e. the harmonic oscillators in three and one dimensions) and for special (symmetric, i.e. $b=0$) cases of PI and PII potentials. A common feature of these algebras is that, in contrast with the case of potential algebras, the variable transformation now results in an $h(x)$ function which is different from a constant, and consequently the Hamiltonian has more complicated structure (see (3.12)) and does not necessarily commute with the generators. Due to the same circumstances, the SUSYQM ladder operators differ from J_+ and J_- in this case.

The potentials and algebras obtained in the PI and PII cases are practically the same in this case, due to the $b = 0$ restriction. Similarly to the potential algebra, the spectrum-generating algebra can also be compact ($su(2)$) or non-compact ($su(1,1)$), and this fact is reflected in the number of bound states.

The eigenstates of the three-dimensional harmonic oscillator belonging to the same angular momentum l also form a basis for the irreducible representations of the $SU(1,1)$ group generated by the $su(1,1)$ algebra. However, in this approach there are no operators connecting states with different l, which could be used to enlarge $SU(1,1)$ into a larger dynamical group [4]. Similarly, the two $SU(1,1)$ unitary irreducible representations to which negative- and positive-parity states of the one-dimensional harmonic oscillator belong, are also disjoint in this sense: there are no operators in this approach which would ladder between states with different parity. We note, however, that the SUSYQM ladder operators do connect states of different l (in the LI case) and parity (in the HI case), but they do not form a potential algebra. They are, actually similar to some elements of the $\mathcal{G}(0,1)$ algebra described by Miller [3] in connection with the C′- and D′-type factorizations. In the latter case they can also be recognized as the familiar raising and lowering operators, $a^{\dagger}$ and a, creating and annihilating one quantum of the one-dimensional harmonic oscillator. One peculiarity of the $SU(1,1)$ irreducible representations related to the three- and one-dimensional harmonic oscillators is that j and m are quarter-integers, rather than integers or half-integers. This finding is similar to some earlier results [18,38].

It is remarkable that the $sl(2,R) \simeq su(1,1)$ algebra has also been identified as the spectrum-generating algebra of the one-dimensional harmonic oscillator in a study which originally aimed at the determination of the 'maximal kinematical invariance group' (MKI) of the harmonic oscillator, i.e. that of the largest group of coordinate (including time) transformations leaving invariant the corresponding Schrödinger equation [42]. In this context the auxiliary phase variable ϕ can be related to the time variable. A similar analysis of some other potentials has also been carried out [43], and the results have been generalized to the formalism of SUSYQM [44] as well. Apart from the harmonic oscillator, however, the algebraic structures (MKIs) found in this way can not be linked to our realization of the $su(1,1)$ algebra.

We have identified a potential algebra and/or a spectrum-generating algebra for eight of the twelve trivial shape-invariant potentials. In case of the PII class (or type E [2,3]) potentials such algebras could only be constructed for the symmetric ($b = 0$) case in this approach, while the Coulomb problem in three dimensions, i.e. the LII (or type F [2,3]) class turned out to be inaccessible for the present study. These two classes are related to $\mathcal{T}_6$ in Miller's approach [3], which is the Lie algebra of the Euclidean group in three dimensions. This author discusses two more factorization types (C″ and D″) with the underlying $\mathcal{G}(0,0)$ Lie algebra, but these do not represent shape-invariant potentials, rather they correspond to free motion in three and one dimension, and are related to the Bessel and exponential functions.

The algebras discussed here can be embedded into some larger algebras. This is the case with the PI($i\sinh(ax)$), PI($\cosh(ax)$) and LIII potentials, where the $so(2,2)$, $iso(2,1)$ and $so(3,1)$ dynamical potential algebras have been identified [26]. The PI($\cosh(2ax)$) potential is related to the $SO(2,2)$ potential group [13,14], while the LI potential, i.e. the three-dimensional harmonic oscillator, has $N(3) \otimes Sp(6,R)$ or $SU(3,1)$ as possible dynamical groups [4].

The present work can be generalized in various directions. Here we have considered only one possible differential realization of the $SU(1,1)$ generators. Other realizations of this algebra are also known. Cordero *et al* [12] considered a Hamiltonian which is a linear (rather than quadratic) form of the generators. Examples for a similar differential realization of the $su(1,1)$ algebra can also be found in [4, 25, 25, 45], where potentials related to the generalized Laguerre polynomials have been discussed. Miller [3] also proposed an alternative realization of the algebras $\mathcal{G}(1,0)$, $\mathcal{G}(0,1)$ and $\mathcal{G}(0,0)$ in terms of one variable (rather than two variables). A similar systematic study of these approaches could also be worthwhile, as it could probably reveal new algebraic structures related to solvable potentials. Considering larger algebras from the beginning and applying the same procedure to them could also be instructive.

Finally, generalizing the results to non-shape-invariant potentials could help us to relate algebraic structures to some less well known potentials. This would require more general variable transformations $y \to x(y)$ of the generators. Starting from the differential equation of the Gegenbauer polynomials for example, a transformation of this kind recovers the Ginocchio potential [36] associated with an $su(1,1)$ potential algebra [22]. However, as we have already mentioned, this procedure does not guarantee the existence of an $su(1,1)$ potential algebra or any other algebra related to the transformed potential, as in certain circumstances the transformations may destroy the original algebraic structure. Nevertheless, our approach helps to select those more general transformations which preserve the $su(1,1)$ algebra. Therefore a systematic study of these non-shape-invariant potentials seems to be necessary to decide which of their subclasses can be related to algebraic structures in our approach.

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